Fast and accurate methods for the discretization of singular integral operators given on surfaces

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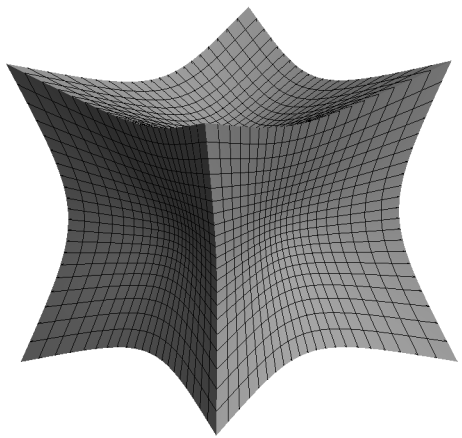
March 15, 2018

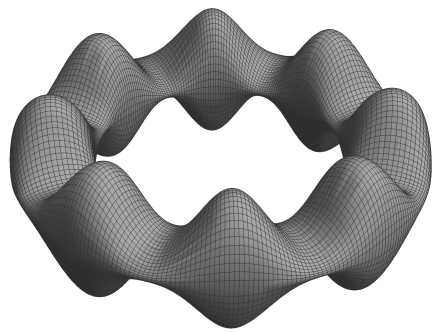
This is joint work with Zydrunas Gimbutas (NIST Boulder)

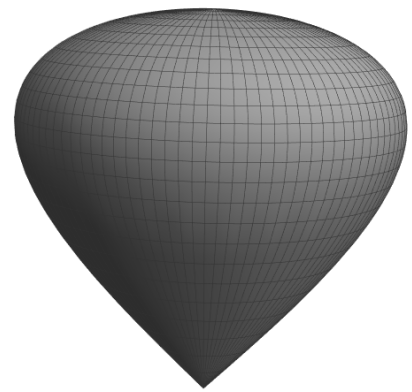
Linear elliptic boundary value problems

The work I will describe today is aimed at finding high-accuracy solutions of certain linear elliptic boundary value problems given on domains with complicated geometry.

For instance:







$$\begin{cases} \Delta u(x) + k^2 u(x) = f(x) \quad \text{on} \quad \Omega \\ \dfrac{\partial u}{\partial \nu}(x) = g(x) \quad \text{on} \quad \partial\Omega \end{cases}$$

Methods for solving such problems

Perhaps the most obvious way to solve a linear elliptic boundary value problem is via a global spectral scheme.

Many boundary value problems of the form

$$\begin{cases} a^{ij}(x)D_iD_ju(x) + b^i(x)D_iu(x) + c(x)u(x) = f(x) \quad \text{on} \quad \Omega \\ u(x) = g(x) \quad \text{on} \quad \partial\Omega \end{cases}$$

give rise to well-conditioned, invertible operators which act

$$H^2(\Omega) \to L^2(\Omega) \oplus V^{3/2}(\partial\Omega),$$

where $V^{3/2}(\partial\Omega)$ is the trace space of $H^2(\Omega)$.

Straightforward global spectral schemes yield high accuracy. Indeed, this operator is a compact perturbation of a coercive operator, with the consequence that approximate solutions of it obtained via Galerkin discretization converge "quasioptimally."

Such an approach is quite slow, however.

Methods for solving such problems

Spectral element methods, discontinuous Galerkin methods and finite element methods can be applied to boundary value problems of the type

$$\begin{cases} a^{ij}(x)D_iD_ju(x) + b^i(x)D_iu(x) + c(x)u(x) = f(x) \quad \text{on} \quad \Omega \\ u(x) = g(x) \quad \text{on} \quad \partial\Omega. \end{cases}$$

They proceed by subdividing the domain (usually into regular pieces like triangles and rectangles) and representing the restriction of the solution to each piece using a local basis (usually polynomials). There are many variations.

Such methods have many advantages, but they often lead to poorly conditioned discretizations, with the consequence that the accuracy obtainable via such methods is limited.

The essential difficulty is that the solution of a spatially global problem (an elliptic boundary value problem) is represented in a spatially local fashion.

Integral equation methods

Another obvious approach is to compose the differential operator on the right with a parametrix. This leads to an integral equation method.

For instance, we could assume that the solution u of

$$\begin{cases} \Delta u(x) + q(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{cases}$$

is of the form

$$u(x) = \int_\Omega K(x,y)\psi(y)\, dy + \int_{\partial\Omega} \tilde{K}(x,y)\sigma(y)\, dS(y)$$

with

$$K(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|} \quad \text{and} \quad \tilde{K}(x,y) = \frac{\partial}{\partial \nu_y} K(x,y) = \frac{1}{4\pi}\frac{(y-x)\cdot \nu_y}{|x-y|^3}.$$

Integral equation methods

This results in a system of integral equations that looks more-or-less like this:

$$\psi(x) + q(x) \int_\Omega K(x,y)\psi(y)\, dy + q(x) \int_{\partial\Omega} \tilde{K}(x,y)\sigma(y)\, dS(y) = f(x) \quad \text{for all} \quad x \in \Omega$$

$$\sigma(x) + q(x) \int_\Omega K(x,y)\psi(y)\, dy + q(x) \int_{\partial\Omega} \tilde{K}(x,y)\sigma(y)\, dS(y) = g(x) \quad \text{for all} \quad x \in \partial\Omega$$

The principal advantage of this approach is the resulting operators are bounded on spaces of square integrable functions. Among other things, this means that:

- the unknowns ψ and σ can be represented via local expansions without penalty (that is, without increasing the condition number of the discretized system of linear equations much beyond that of the underlying elliptic operator). As a consequence, very high accuracy can typically be obtained.
- there are very few conditions on the mesh used to represent the unknown functions because there is no need to match their values at the artificial boundaries which are introduced by decomposing the domain

In addition, in some cases, the dimension of the problem can be reduced by one.

Disadvantages of integral equation methods

The principal disadvantage of such methods is already evident: one must have sufficient analytic understanding of the boundary value problem under consideration to select appropriate integral kernels and evaluate them.

Moreover, the coefficient matrices of the systems of linear equations which arise from integral equation methods are dense.

A third, perhaps less obvious, disadvantage is that integral equation methods require the discretization of singular integral operators given on volume regions, curves and surfaces.

Discretization of singular integral operators

The case of volume integrals is a mostly solved problem.

> *Greengard, et. al.: break them up into small, regularly-shaped pieces and rely on absurdly small per piece processing times.*

The discretization of boundary integral operators on smooth curves is also a (mostly) solved problem.

> *Too may authors to mention; generalized Gaussian quadratures for logarithmic singularities and principal value integrals; finite parts integrals via regularization; etc.*

The discretization of boundary integral operators given on curves with corners is also no trouble.

> *Rokhlin-Serkh: explicit formulas for solutions on domains with wedges which leads to efficient quadrature rules for general domains with corners.*

Singular integral operators given on surfaces

The discretization of the singular integral operators given on surfaces is more vexing.

I will describe a scheme for the discretization of those operators which arise from the Dirichlet and Neumann boundary value problems for the Helmholtz equation. That is, the problems

$$\begin{cases} \Delta u(x) + k^2 u(x) = f(x) & \text{on} \quad \Omega \\ u(x) = g(x) & \text{on} \quad \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta u(x) + k^2 u(x) = f(x) & \text{on} \quad \Omega \\ \dfrac{\partial u}{\partial \nu}(x) = g(x) & \text{on} \quad \partial\Omega \end{cases}$$

Here, Ω is a surface whose boundary is $\partial\Omega$ and k will be a relatively small real-valued constant.

Singular integral operators given on surfaces

The relevant layer potentials are

$$S[\sigma](x) = \int_{\partial\Omega} K(x,y)\ \sigma(y)\ ds(y)$$

and

$$D[\sigma](x) = \int_{\partial\Omega} \frac{\partial}{\partial\eta_y} K(x,y)\ \sigma(y)\ ds(y)$$

with

$$K(x,y) = \frac{1}{4\pi} \frac{\exp(ik|x-y|)}{|x-y|}.$$

I will use the notation S' and D' to denote further differentiation with respect to the outward pointing unit normal. For example:

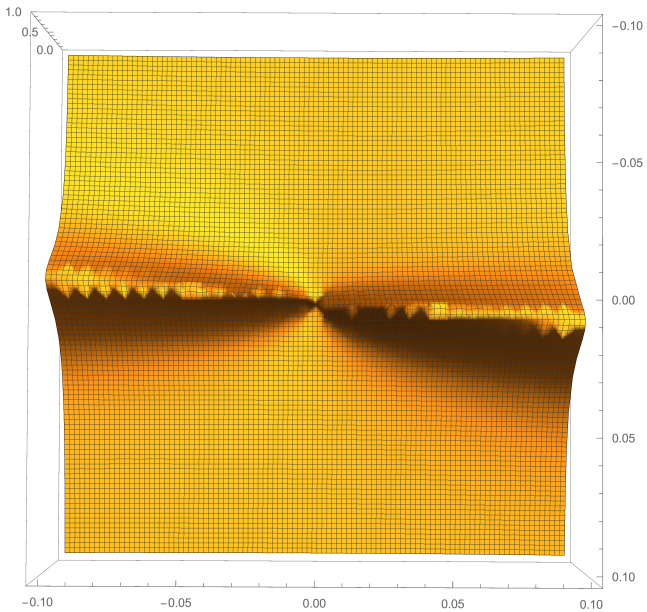
$$D'[\sigma](x) = \int_{\partial\Omega} \frac{\partial}{\partial\eta_x} \frac{\partial}{\partial\eta_y} K(x,y)\ \sigma(y)\ ds(y)$$

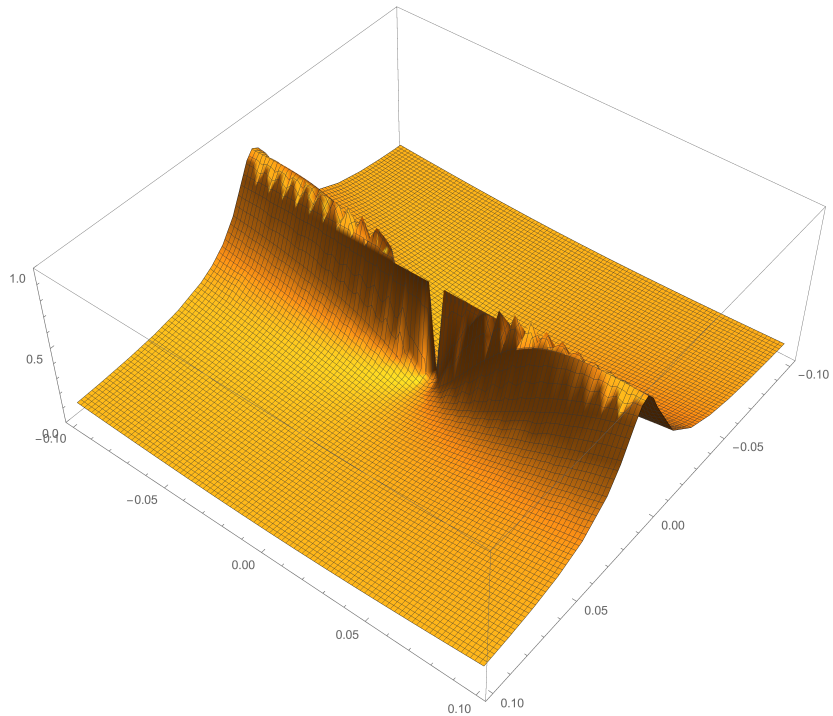
Singular integral operators given on surfaces

These kernels have singularities which are integer powers of

$$\frac{1}{|\psi(x) - \psi(y)|},$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ is a parameterization of the surface $\partial\Omega$. Below are a few graphs of the function $\dfrac{|y|}{|\psi(y)|}$, when $\psi(y_1, y_2) = \begin{pmatrix} t - 10s \\ t + 2s \\ t/5 \end{pmatrix}$.





Who's afraid of singular integrals?

Quadrature by expansion (QBX) is an elegant scheme for discretizing singular integral operators of the form

$$S[\sigma](x) = \int_{\partial\Omega} K(x,y)\ \sigma(y)\ ds(y)$$

without explicitly evaluating a large number of singular integrals of this type.

The basic idea is to construct an expansion of $S[\sigma](x)$ around a point x which is bounded away from $\partial\Omega$, and then use this expansion to evaluate S at points on the boundary.

There is a large and rapidly expanding literature on these methods; see, for instance, the papers of Köckner, Barnett, Greengard, O'Neil, Tornberg, Klinteberg, Rachh, Siegel, others?

Our scheme, by comparison, proceeds by evaluating the singular integrals which arise in a brutal, simple-minded fashion.

Representation of the surface $\partial\Omega$

$$S[\sigma](x) = \int_{\partial\Omega} K(x,y)\ \sigma(y)\ ds(y)$$

We assume that $\partial\Omega$ is described via a collection of smooth mappings given on triangles.

That is, the user provides us with a collection $\{T_j\}$ of triangles in the plane and, for each triangle T_j, also specifies a smooth mapping

$$\rho_j : T_j \to \mathbb{R}^3.$$

- The sets $\rho_j(T_j)$ should form a disjoint cover of $\partial\Omega$.
- This framework includes the case of triangulated surfaces but can incorporate higher order information if it is available.
- Piecewise smooth surfaces can be described with this framework.

Representation of the argument σ

$$S[\sigma](x) = \int_{\partial\Omega} K(x,y)\ \sigma(y)\ ds(y)$$

For each j, we assume that the function

$$\sigma(\rho_j(t))$$

given on the triangle T_j is a polynomial of degree N. We represent this function through its values at the nodes of a quadrature which integrates polynomials of degree $2N$. We call this the discretization quadrature.

To be precise, we represent the restriction of σ to T via the vector of scaled values

$$\begin{pmatrix} \sigma(\rho_j(t_1))\sqrt{|d\rho_j(t_1)|}\sqrt{w_1} \\ \sigma(\rho_j(t_2))\sqrt{|d\rho_j(t_2)|}\sqrt{w_2} \\ \vdots \\ \sigma(\rho_j(t_k))\sqrt{|d\rho_j(t_k)|}\sqrt{w_k} \end{pmatrix}$$

where $d\rho_j(t)$ denotes the Jacobian matrix of the parameterization ρ_j at t, $|d\rho_j(t)|$ is the determinate of that matrix and $t_1, \ldots, t_k, w_1, \ldots, w_k$ is the quadrature rule.

Representation of the argument σ

$$S[\sigma](x) = \int_{\partial\Omega} K(x,y)\ \sigma(y)\ ds(y)$$

For each j, we assume that the function

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The norm of this vector is the $L^2(\rho_j(T_j))$ norm of the restriction of σ to the surface region $\rho_j(T_j)$.

N	quadrature order	nodes
4	8	17
6	12	32
8	16	52
10	16	82
12	24	112
16	32	192

Lengths of quadrature rules used in my discretization code.

Definition of the discretization of S

The discretization of the operator S is the square matrix which maps:

$$\begin{pmatrix} \sigma(\rho_1(t_1))\sqrt{|d\rho_1(t_1)|}\sqrt{w_1} \\ \vdots \\ \sigma(\rho_1(t_k))\sqrt{|d\rho_1(t_k)|}\sqrt{w_k} \\ \sigma(\rho_2(t_1))\sqrt{|d\rho_2(t_1)|}\sqrt{w_1} \\ \vdots \\ \sigma(\rho_2(t_k))\sqrt{|d\rho_2(t_k)|}\sqrt{w_k} \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} S[\sigma](\rho_1(t_1))\sqrt{|d\rho_1(t_1)|}\sqrt{w_1} \\ \vdots \\ S[\sigma](\rho_1(t_k))\sqrt{|d\rho_1(t_k)|}\sqrt{w_k} \\ S[\sigma](\rho_2(t_1))\sqrt{|d\rho_2(t_1)|}\sqrt{w_1} \\ \vdots \\ S[\sigma](\rho_2(t_k))\sqrt{|d\rho_2(t_k)|}\sqrt{w_k} \\ \vdots \end{pmatrix}$$

Because of our choice of quadrature and the presence of square root weighting, this is equivalent to a standard Galerkin method, and inherits all of the nice properties of such methods (e.g., quasioptimal convergence).

The basic computational unit for our solver is a routine which evaluates a block of the discretization matrix corresponding to a single target node ξ and a single source triangle T. Assuming ρ is the parameterization given on T and w is the quadrature weight associated with ξ, this block is the $1 \times k$ matrix which maps

$$\begin{pmatrix} \sigma(\rho(t_1))\sqrt{d\rho(t_1)}\sqrt{w_1} \\ \vdots \\ \sigma(\rho(t_k))\sqrt{d\rho(t_k)}\sqrt{w_k} \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{w}\int_T K(\xi, \rho(s))\sigma(\rho(s))\,|d\rho(s)|\,ds \end{pmatrix}$$

Singular integral operators given on surfaces

We form this block by first constructing a quadrature rule

$$\int_T K(\xi, \rho(s))\sigma(\rho(s))\,|d\rho(s)|\,ds \approx \sum_{j=1}^{m} K(\xi, \rho(s_j))\sigma(\rho(u_j))\,|d\rho(u_j)|\,v_j.$$

Next, we form the vector

$$\begin{pmatrix} K(\xi, \rho(s_1))\sqrt{v_1}\sqrt{w}\sqrt{|d\rho(s_1)|} \\ K(\xi, \rho(s_2))\sqrt{v_2}\sqrt{w}\sqrt{|d\rho(s_2)|} \\ \vdots \\ K(\xi, \rho(s_m))\sqrt{v_m}\sqrt{w}\sqrt{|d\rho(s_m)|} \end{pmatrix}$$

and apply the interpolation matrix which maps

$$\begin{pmatrix} \sigma(\rho(t_1))\sqrt{|d\rho(t_1)|}\sqrt{w_1} \\ \sigma(\rho(t_2))\sqrt{|d\rho(t_2)|}\sqrt{w_2} \\ \vdots \\ \sigma(\rho(t_k))\sqrt{|d\rho(t_k)|}\sqrt{w_k} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma(\rho(s_1))\sqrt{|d\rho(s_1)|}\sqrt{v_1} \\ \sigma(\rho(s_2))\sqrt{|d\rho(s_2)|}\sqrt{v_2} \\ \vdots \\ \sigma(\rho(s_m))\sqrt{|d\rho(s_m)|}\sqrt{v_m} \end{pmatrix}$$

to its right-hand side.

Singular integral operators given on surfaces

All that remains is the construct the appropriate quadrature rules for the integral

$$\int_T K(\xi, \rho(s))\sigma(\rho(s))\,|d\rho(s)|\;\; ds,$$

where

- T is a triangle
- $\rho : T \to \mathbb{R}^3$ is a smooth mapping
- $d\rho(s)$ denotes the Jacobian matrix of ρ at the point s;
- ξ is a point on $\partial\Omega$.

There are three regimes, depending on the location of ξ relative to $\rho(T)$:

- far regime: ξ is "far" from $\rho(T)$.
- nearly singular regime: ξ is "close" to, but not in, $\rho(T)$;
- singular regime: ξ is inside of $\rho(T)$;

Trivial scheme for far-field interactions

When ξ is far from the surface region $\rho(T)$, the integral

$$\int_T K(\xi, \rho(s))\sigma(\rho(s))\,|d\rho(s)|\;\; ds$$

can be evaluated via the discretization quadrature on T used to represent σ. Thus the interpolation matrix is the identity in this case, and the corresponding entries of the discretization matrix are simply of the form

$$K(\xi, \rho(x_j))\sqrt{w}\sqrt{w_j}.$$

Obviously, we do not actually apply any interpolation matrices in this regime.

Thankfully, this is the most commonly occurring case.

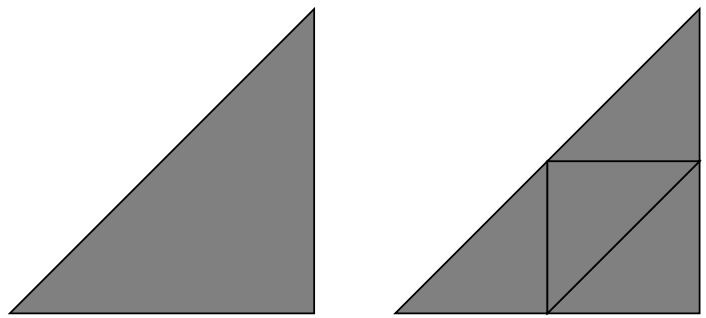
Adaptive scheme for near interactions

In order to evaluate

$$\int_T K(\xi, \rho(y))\sigma(\rho(y))\,|d\rho(y)|$$

with ξ near T, we recursively divide T into a disjoint collection of triangles $T_1, \ldots, T_M$ until the point ξ is in the far-field of every T_j.

We interpolate the values of σ from the discretization quadrature nodes on T to the discretization quadrature nodes on each of the triangles T_j and then evaluate each of the integrals



$$\int_{T_j} K(\xi, \rho(y))\sigma(\rho(y))\,|d\rho(y)|$$

using the discretization quadratures.

This is surprisingly efficient since each division corresponds to the application of the same $4k \times k$ matrix.

The singular regime

This is the least frequently occurring regime, but the most challenging. Almost all existing methods fail to accurately evaluate these integrals, except when $\partial\Omega$ is extremely simple.

Changing to polar coordinates takes

$$\int_T K(\xi, \rho(s))\sigma(\rho(s))\,|d\rho(s)|\,dy$$

into an integral of the form

$$\int_0^{2\pi}\int_0^{R(\theta)} \Big(q_{-1}(\theta) + q_0(\theta)r + q_1(\theta)r^2 + q_2(\theta)r^3 + \cdots\Big)\,dr\,d\theta$$

where $R(\theta)$ is the parameterization of the boundary of the integration domain T in polar coordinates and the $q_j(\theta)$ are periodic and analytic in some strip containing the real axis.

Exponential convergence can be obtained by dividing the outer integral into regions on which $R(\theta)$ is smooth and applying tensor product Gaussian quadratures.

Difficulties with this standard approach

The integrand

$$q_{-1}(\theta) + q_0(\theta)r + q_1(\theta)r^2 + q_2(\theta)r^3 + \cdots$$

is analytic on a strip containing the real line but each q_j is of the form

$$q_j(\theta) = \frac{r_j(\theta)}{[I(\theta)]^{j+2}},$$

where r_j is a trigonometric polynomial of **finite order** (which depends on j) but $I(\theta)$ can have zeros close to the real axis.

In fact, if $\{\xi_1, \xi_2\}$ is the basis of the tangent space to the surface at the target node induced by the parameterization ρ, then the zeros of I are the solutions w of the equation

$$\cot(w) = -\lambda^{-1} \exp(\pm i\eta)$$

where

$$\cos(\eta) = \frac{\xi_1 \cdot \xi_2}{|\xi_1|\,|\xi_2|} \quad \text{and} \quad \lambda = \frac{|\xi_1|}{|\xi_2|}.$$

See, for instance, Wendland & Schwab.

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where r_j is a trigonometric polynomial of **finite order** (which depends on j) but $I(\theta)$ can have zeros close to the real axis.

That is, the proximity of the poles of the functions q_j to the real axis is a measure of how nonconformal the mapping ρ is at the target node ξ. The less conformal (more stretched out), the closer the poles.

Difficulties with this standard approach

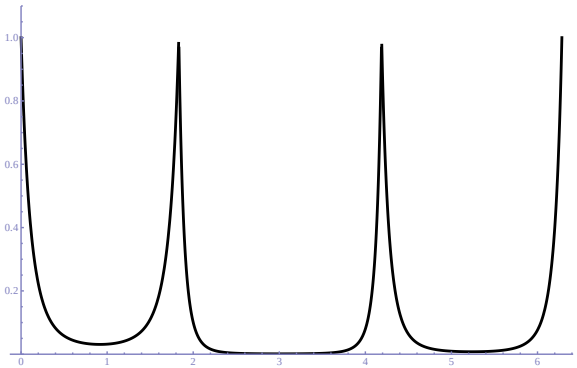
The outer integral in

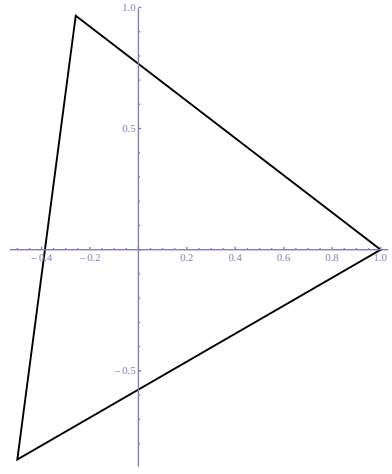
$$\int_0^{2\pi} \int_0^{R(\theta)} \left(q_{-1}(\theta) + q_0(\theta) r + q_1(\theta) r^2 + q_2(\theta) r^3 + \cdots \right) dr\, d\theta$$

is of the form

$$\int_0^{2\pi} \left(q_{-1}(\theta) R(\theta) + q_0(\theta) \frac{(R(\theta))^2}{2} + q_1(\theta) \frac{(R(\theta))^3}{3} + q_2(\theta) \frac{(R(\theta))^4}{4} + \cdots \right) d\theta$$

and the functions $(R(\theta))^k$, while analytic on the real line, are poorly behaved in their own right.





Summary

Exponential convergence is obtained by this standard method. That is, the error in the approximation of the integral obtained using m-point product Legendre quadratures behaves as

$$C \exp\left(-\lambda \cdot m\right).$$

But, a number of circumstances can cause the constant λ to be quite small. For instance, λ is small when:

- The mapping ρ is moderately nonconformal at the target node ξ.
- The target node is close to the boundary of the parameterization domain T.
- The parameterization domain T is "stretched out."

Simplification of the integrands

Observation:

Inducing conformality at the target node simplifies the integrand.

Indeed, when the parameterization ρ is conformal at the target node x, the function $I(\theta)$ becomes a constant. In this case, the kernel admits representation as

$$K(\rho(x), \rho(x + (r\cos(\theta), r\sin(\theta)))) = \frac{q_{-1}(\theta)}{r} + q_0(\theta) + q_1(\theta)r + q_2(\theta)r^2 + \cdots$$

where the q_j are **trigonometric polynomials of finite order**.

NOTE: The parameterization ρ only needs to be conformal at one point; it does **not** need be conformal on its whole domain.

Not enough

Inducing conformality at the target node by modifying the parameterization (i.e., by stretching out the triangle T) greatly simplifies the integrand.

But it does nothing to make the evaluation of the outer integral

$$\int_0^{2\pi} \left(q_{-1}(\theta) R(\theta) + q_0(\theta) \frac{R(\theta)^2}{2} + q_1(\theta) \frac{R(\theta)^3}{3} + q_2(\theta) \frac{R(\theta)^4}{4} + \cdots \right) d\theta$$

simpler.

Indeed, it often makes the evaluation of this integral **more difficult.** In many cases, the modified triangle is more stretched out than the original triangle T and this worsens the behavior of the function $R(\theta)$.

Solution: **Simple-minded brutality**

We precompute table of quadrature rules which allows for the efficient evaluation of the integrals

$$\int_0^{2\pi} \left(q_{-1}(\theta) R(\theta) + q_0(\theta) \frac{R(\theta)^2}{2} + q_1(\theta) \frac{R(\theta)^3}{3} + q_2(\theta) \frac{R(\theta)^4}{4} + \cdots \right) d\theta$$

where R is the parameterization any triangle — no matter how stretched.

Generalized quadrature

We can construct a quadrature for a collection of user-specified functions

$$f_1(x),\ f_2(x),\ f_3(x), \ldots, f_n(x) \tag{1}$$

on an interval $[a, b]$ by producing a **sparse** solution $x_1, \ldots, x_m, w_1, \ldots, w_m$ to the nonlinear system of equations

$$\sum_{j=1}^{m} f_i(x_j) w_j = \int_a^b f_i(x) dx, \quad i = 1, \ldots, n.$$

Ideally, m would be equal to $k/2$, where k is the numerical rank of the collection of the functions (1).

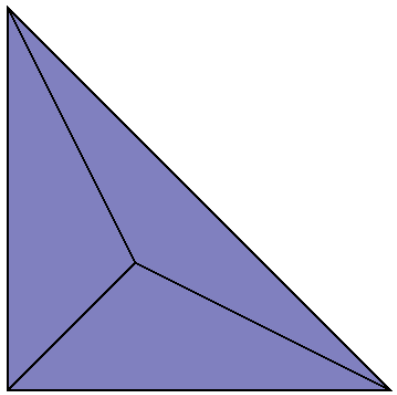
In contrast to most quadrature techniques, the number of points in generalized Gaussian quadratures depend on the dimension of space of integrands and not on their behavior.

An algorithm for constructing these formula in a stable and efficient fashion can be found in:

B—, Rokhlin and Gimbutas, "A nonlinear optimization procedure for generalized Gaussian quadratures." *SIAM Journal of Scientific Computing* 32 (2010), pp. 1761-1788.

Generalized quadrature

We proceed by dividing the triangle T into three pieces by connecting the target node to the vertices and treating each piece separately.



The effect of this action is to reduce the number of parameters we will need.

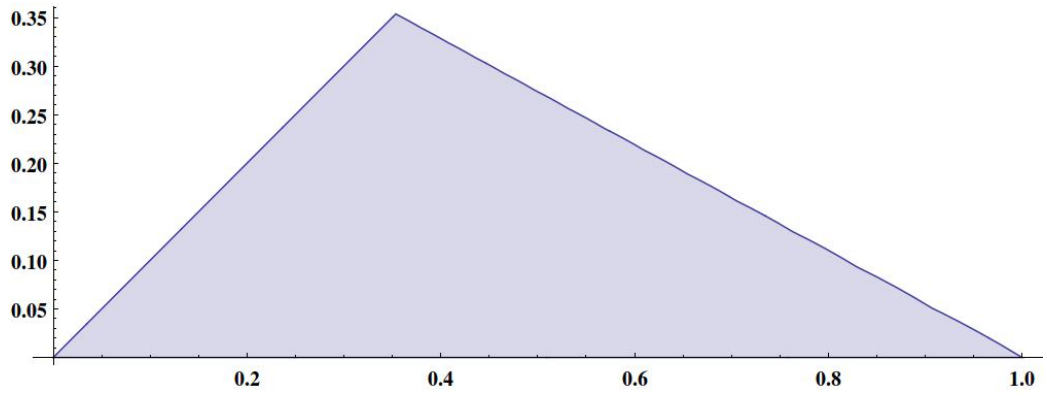
Each of the resulting triangles can be transformed into a triangle with vertices

$$(0,0), \quad (1,0) \quad \text{and} \quad (\alpha\cos(\phi), \alpha\sin(\phi)).$$

without effecting the representation of the kernel underlying this approach.

Generalized quadrature

$$R_{\alpha,\phi}(\theta) = \frac{\alpha \sin(\theta)}{\sin(\theta) - \alpha \sin(\theta - \phi)}$$



We wish to evaluate integrals of the form

$$\int_0^{\phi} \int_0^{R_{\alpha,\phi}(\theta)} \left(p_{-1}(\theta) + p_0(\theta) r + p_1(\theta) r^2 + \ldots \right) dr \; d\theta.$$

The inner integral can be evaluated efficiently via a Legendre quadrature; so we focus on the outer integral, which can be rewritten in the form

$$\int_0^1 \left(p_{-1}(\phi\theta) R_{\alpha,\phi}(\phi\theta) + p_0(\phi\theta) \frac{R^2_{\alpha,\phi}(\phi\theta)}{2} + \cdots + p_N(\phi\theta) \frac{R^{N+2}_{\alpha,\phi}(\phi\theta)}{N+2} \right) \phi d\theta$$

Each quadrature formula is designed to handle integrals of this type for a fixed range of the parameters α and ϕ and a fixed value of the integer N.

We call N the order of the formula.

Principal value and finite parts integrals

Modifications of this scheme allow for the evaluation of principal value integrals such as the tangential derivatives of the

$$S\left[\sigma\right](x) = \int_{\partial\Omega} K(x,y)\ \sigma(y)\ ds(y)$$

and for Hadamard finite parts integrals such as

$$D'\left[\sigma\right](x) = \int_{\partial\Omega} \frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y} K(x,y)\ \sigma(y)\ ds(y).$$

In particular, we can use this scheme to compute the Dirichlet-to-Neumann operator on complicated surfaces (although our scheme for doing so loses many digits due to the ill-conditioning).

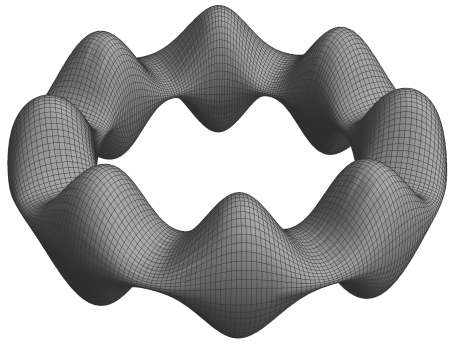
Laplace's equation on tori

$$\Delta u = 0 \quad \text{in } \Omega^c$$

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega$$

α	N_{tri}	N	N_{self}	T_{self}	T_{near}	T_{mult}	T_{total}	E
1.00	4	612	983	1.43×10^{-01}	$1.11 \times 10^{+00}$	2.18×10^{-01}	$1.60 \times 10^{+00}$	1.36×10^{-04}
	16	2448	978	5.83×10^{-01}	$2.84 \times 10^{+00}$	$3.44 \times 10^{+00}$	$7.86 \times 10^{+00}$	1.48×10^{-10}
	64	9792	976	$2.57 \times 10^{+00}$	$8.64 \times 10^{+00}$	$4.01 \times 10^{+01}$	$6.31 \times 10^{+01}$	8.64×10^{-14}
0.25	16	2448	976	5.60×10^{-01}	$1.63 \times 10^{+00}$	$3.23 \times 10^{+00}$	$6.31 \times 10^{+00}$	7.07×10^{-07}
	64	9792	976	$2.26 \times 10^{+00}$	$6.41 \times 10^{+00}$	$2.77 \times 10^{+01}$	$4.45 \times 10^{+01}$	2.19×10^{-11}
	256	39168	976	$8.88 \times 10^{+00}$	$2.54 \times 10^{+01}$	$1.38 \times 10^{+02}$	$2.19 \times 10^{+02}$	2.85×10^{-14}
0.10	40	6120	976	$1.42 \times 10^{+00}$	$3.34 \times 10^{+00}$	$1.09 \times 10^{+01}$	$1.92 \times 10^{+01}$	5.88×10^{-07}
	160	24480	975	$5.57 \times 10^{+00}$	$1.45 \times 10^{+01}$	$7.70 \times 10^{+01}$	$1.16 \times 10^{+02}$	1.11×10^{-11}
	640	97920	976	$2.23 \times 10^{+01}$	$5.91 \times 10^{+01}$	$5.00 \times 10^{+02}$	$6.92 \times 10^{+02}$	2.30×10^{-14}
0.01	400	61200	975	$1.38 \times 10^{+01}$	$2.86 \times 10^{+01}$	$1.05 \times 10^{+02}$	$1.79 \times 10^{+02}$	5.12×10^{-07}
	1600	244800	975	$5.54 \times 10^{+01}$	$1.32 \times 10^{+02}$	$8.47 \times 10^{+02}$	$1.20 \times 10^{+03}$	8.39×10^{-12}
	6400	979200	975	$2.22 \times 10^{+02}$	$5.94 \times 10^{+02}$	$4.76 \times 10^{+03}$	$6.70 \times 10^{+03}$	7.07×10^{-13}

Convergence study on a deformed torus



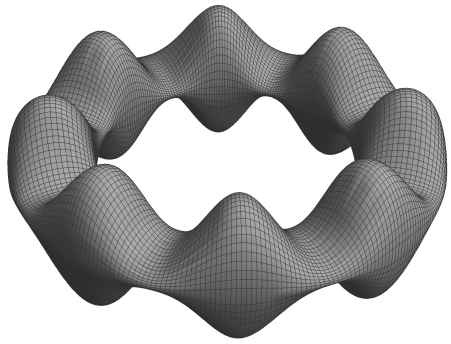
$$\Delta u + k^2 u = 0 \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

Approximately 3 wavelengths in diameter.

N	Relative L^2 error potential	Relative L^2 error in normal derivative
576	0.289×10^{-01}	0.206×10^{-02}
2304	0.296×10^{-03}	0.191×10^{-02}
9216	0.267×10^{-05}	0.584×10^{-04}
36864	0.418×10^{-08}	0.752×10^{-06}

4^{th} order quadratures

Convergence study on a deformed torus



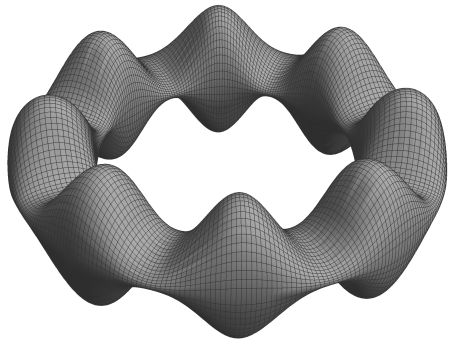
$$\Delta u + k^2 u = 0 \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

Approximately 3 wavelengths in diameter.

N	Relative L^2 error potential	Relative L^2 error in normal derivative
1664	0.829×10^{-03}	0.303×10^{-03}
6656	0.159×10^{-05}	0.173×10^{-05}
26624	0.187×10^{-09}	0.159×10^{-07}
106496	0.480×10^{-13}	0.122×10^{-09}

8^{th} order quadratures

Convergence study on a deformed torus



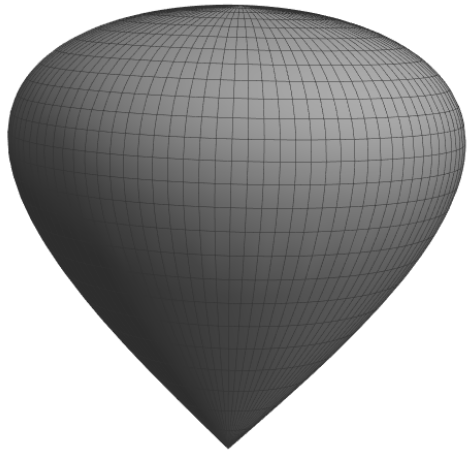
$$\Delta u + k^2 u = 0 \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

Approximately 3 wavelengths in diameter.

N	Relative L^2 error potential	Relative L^2 error in normal derivative
3584	0.800×10^{-04}	0.103×10^{-03}
14336	0.155×10^{-07}	0.484×10^{-07}
57344	0.102×10^{-11}	0.213×10^{-08}
229376	0.957×10^{-13}	0.354×10^{-07}

12^{th} order quadratures

Sound-hard scattering from a snowcone



Approximately 3 wavelengths in diameter.

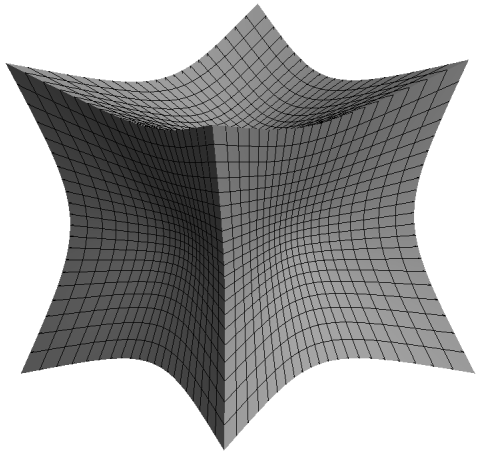
$$\Delta u + k^2 u = 0 \quad \text{in } \Omega^c$$

$$\frac{\partial u}{\partial \eta} = g \quad \text{on } \partial\Omega$$

$$|x| \left(\frac{\partial}{\partial |x|} - ik \right) u(x) \to 0 \text{ as } |x| \to \infty$$

N	κ	E
180	$2.40 \times 10^{+0}$	1.52×10^{-03}
720	$2.42 \times 10^{+0}$	2.42×10^{-05}
2880	$2.44 \times 10^{+0}$	1.04×10^{-07}
11520	$2.45 \times 10^{+0}$	9.09×10^{-10}
46080	$2.45 \times 10^{+0}$	7.04×10^{-13}

Sound-soft scattering from a deformed cube



$$\Delta u + k^2 u = 0 \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

Approximately 8 wavelengths in diameter.

N_{tris}	N	Time	Error
192	32256	$1.23 \times 10^{+02}$	1.23×10^{-08}
432	72576	$4.77 \times 10^{+03}$	3.13×10^{-10}
768	129024	$6.21 \times 10^{+03}$	4.13×10^{-12}
1024	172032	$1.22 \times 10^{+04}$	7.13×10^{-14}